**SECOND YEAR ESSAY**

**INTRODUCTION TO FRACTAL GEOMETRY**

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**INTRODUCTION**

Mathematics used to be concerned mostly with sets and functions which could be further inspected with classical calculus. The sets or functions that are not sufficiently smooth used to have a tendency to be ignored as 'pathological' and deemed not worthy of further research. Indeed, they were regarded as individual curiosities and only rarely were thought of as a class to which a general theory might be applicable.

In recent decades, this attitude has changed. Mathematicians realized that a great deal can be said, and is worth saying, about the features of non-smooth objects. Additionally, irregular sets help us make much better models natural phenomena than do the figures of classical geometry. Fractal geometry provides a basis for the study of such irregular sets.

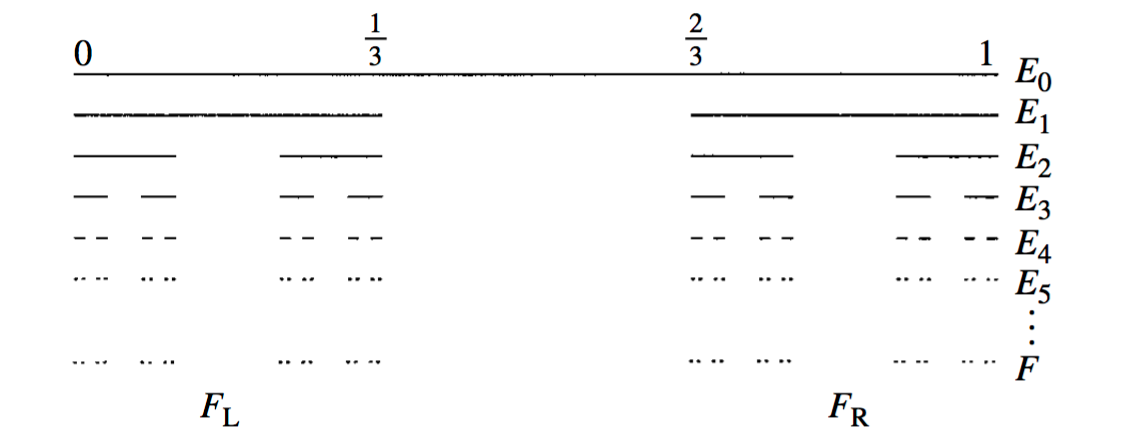
It provides a framework in which a simple process, involving a basic operation repeated many times, can give rise to a highly irregular result. The phrase 'the beauty of fractals' is often heard, a phrase which reflects the unending intricacy of fractal designs alongside the simplicity which underlies their ever-repeating form.

In this essay, I will

* Introduce some elementary fractals
* Show that concepts such as length, area, etc. have limitations when it comes to calculating the “content" of fractals
* Present alternative methods of calculating the "content" of fractals sets
* Discuss a key concept in Fractal geometry called Self-Similarity

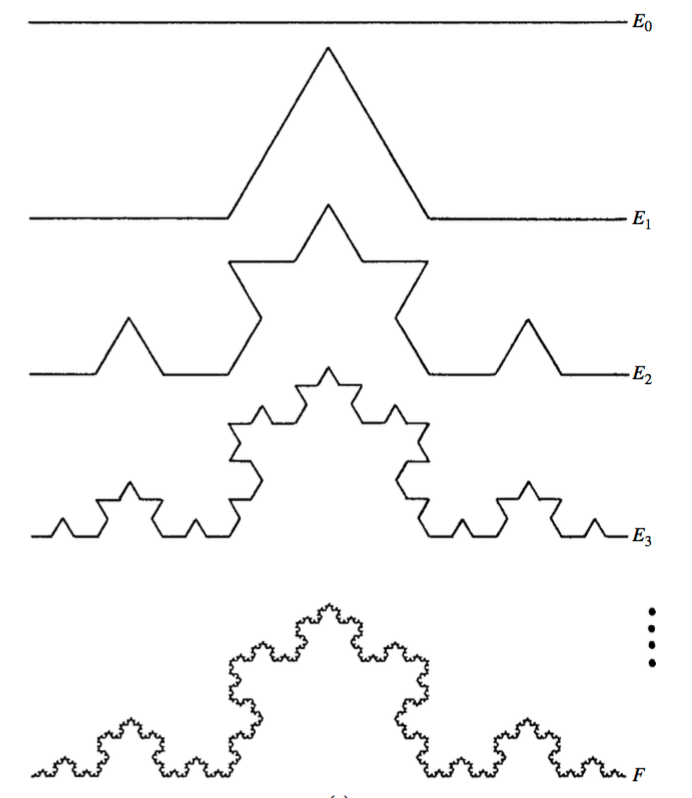
**CANTOR SET**

The middle third Cantor set is a well-known and easily constructed fractals; nevertheless, it has many typical fractal characteristics. It is constructed in the following manner. Let be the interval [0,1]. Delete the middle third and name the new set . Therefore, consists of two intervals [0,1/3] and [2/3. Delete the middle thirds of these intervals and name the new set ; thus comprises the four intervals [0,1/9, [2/9,1/3], [ 2/3,7/9], [ 8/9. We continue this way, consists of intervals each of length . The middle third Cantor set F consists of the numbers that are in for all ; mathematically, . The Cantor set F may be thought of as the limit of the sequences of sets as tends to infinity. [1, Page xvii]

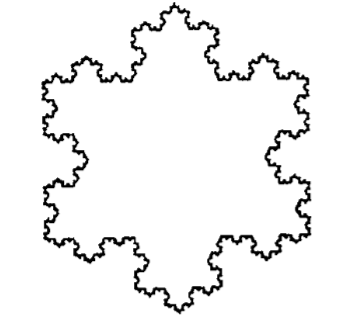


**VON KOCH CURVE**

We let be a line segment of unit length. Remove the middle third of and replace it by the other two sides of the equilateral triangle based on the removed segment. Name this set .We construct by applying the same procedure to each of the segments in , and so on. Thus comes from replacing the middle third of each straight-line segment of by the other two sides of an equilateral triangle. When is large, the curves and differ only in fine detail and as tends to infinity, the sequence of polygonal curves approaches a limiting curve , called the . [1, Page xviii-xix]



Construction of the von Koch curve . At each stage, the middle third of each interval is replaced by the other two sides of an equilateral triangle.



Three von Koch curves fitted together to form a *snowflake curve.*

**Perimeter of the Koch snowflake** - After each iteration, the number of sides of the Koch snowflake increases by a factor of 4, so the number of sides after  iterations is given by

Let the original equilateral triangle have sides of length . Therefore, the length of each side of the snowflake after  iterations is

Finally, we call the perimeter after  iterations is

### Hence, the Koch curve has an infinite perimeter because it increases by a factor with each iteration.

### **Area of the Koch snowflake** - A new triangle is added on each side with each iteration. Therefore, the number of new triangles added in iteration  is

Every new triangle added in an iteration has area equal to one ninth of the area of each triangle added in the previous iteration. Hence, the area of each triangle added in iteration  is

where  is the area of the original triangle. Now we can calculate the total new area added in iteration

The total area of the snowflake after  iterations is

#### **Limits of area** -The limit of the area is

So the area of the Koch snowflake is of the area of the original triangle. Expressed in

terms of the side length *s* of the original triangle this is

. [3]

From this, we can see that the traditional concepts

**FRACTAL DIMENSION**

An integral feature of fractals is their delicate structure, i.e. their detail at arbitrarily small scales. 'Fractal dimension' attempts to quantify this by measuring the rate at which increased detail becomes apparent as we examine a fractal at smaller scales. It is a measure of the complexity of the fractal and of the amount of space it occupies when viewed at high resolution. There are multiple definitions of dimension, but all rely on measuring fractals in some way at finer scales. [2, Page 35]

**Measures -** Measures are an integral part of fractal geometry. One need not be intimidated by the seemingly technical nature of measure theory as for most fractal applications, only a small number of elementary concepts are needed. Additionally, people are often already familiar with these concepts as these are to mass and charge distributions encountered in basic physics.

For now, we will keep our focus on measures on subsets of Basically, a measure gives a numerical 'size' to sets, such that if a set is split into a finite or countable number of pieces reasonably, then the size of the whole is the sum of the sizes of the pieces.

“We call a *measure* on if assigns a non-negative number, possibly , to each subset of such that:

1. ) = 0
2. if B
3. If is countable (or finite) sequence of sets then

and if are disjoint Borel Sets[[1]](#footnote-1).” [Copied from 1, Page 11]

**Hausdorff Measure –** Let be any non-empty subset of -dimensional Euclidean Space, . Then the of is defined as , (the greatest distance between any pair of points in .) If { is a countable (or finite) collection of sets of diameter at most that cover , i.e. with for each , we say that { is cover of .

Suppose that and is a non-negative number. we define

(1)

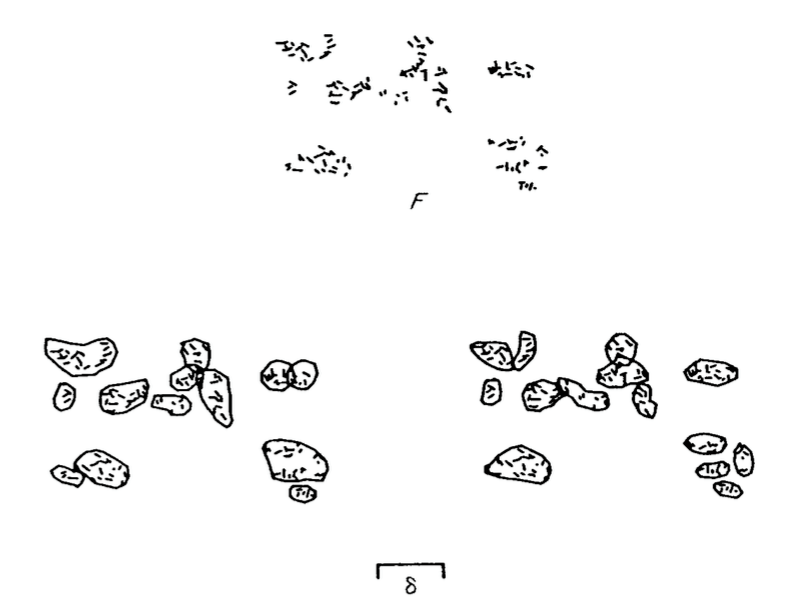
As decreases, the class of permissible covers of in the diagram also decreases. Therefore, the infimum increases, and so approaches a limit as . We write

(2)

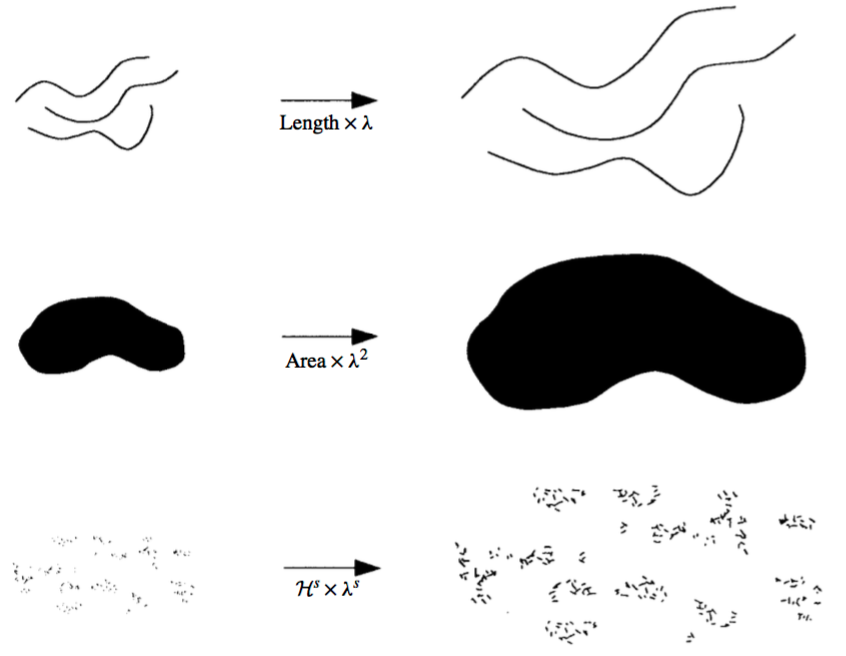
The limit exists for any subset of , though the limiting value can be (and usually is) or . We call the of . With a certain amount of effort, may be shown to be a measure. It is straight forward to show that , that if is contained in then , and that if {} is any countable collection of sets then

It is rather harder to show that there is equality in the above equation if the {} are disjoint Borel sets. [Similar to 1, Page 27-28]

The scaling properties of length, area and volume are well known. On magnification by a factor , the length of a curve is multiplied by , the area of a plane region is multiplied by and the volume of a 3-dimensional object is multiplied by . As might be anticipated, s-dimensional



A set and two possible of . The infimum of over all such {, gives[Taken from 1, Page 28]

  
Scaling sets by a factor increases length by a factor , area by a factor and -dimensional Hausdorff measure by a factor .[Taken from 1, Page 29]

Hausdorff measure scales with a factor (figure above). Such scaling properties are fundamental to the theory of fractals

**Scaling property****–**Let be a similarity transformation of scale factor . If , then

**Proof** *–* If { is a δ-cover of F then is a -cover of , so

on taking the infimum. Letting gives that . Replacing by , and so by , and by gives the opposite inequality required.

A similar argument gives the following basic estimate of the effect of more general transformations on the Hausdorff measures of sets.

**Proposition –** Let and be a mapping that

(3)

For constants and . Then for each

**Proof –** If is a -cover of , then, since , it follows that is and -cover of , where . Thus , so that . As , so , giving .

Condition (3) is known as a *Hölder condition of exponent* ; such a condition implies that is continuous. Particularly important is the case , i.e.

when is called mapping, and

(4)

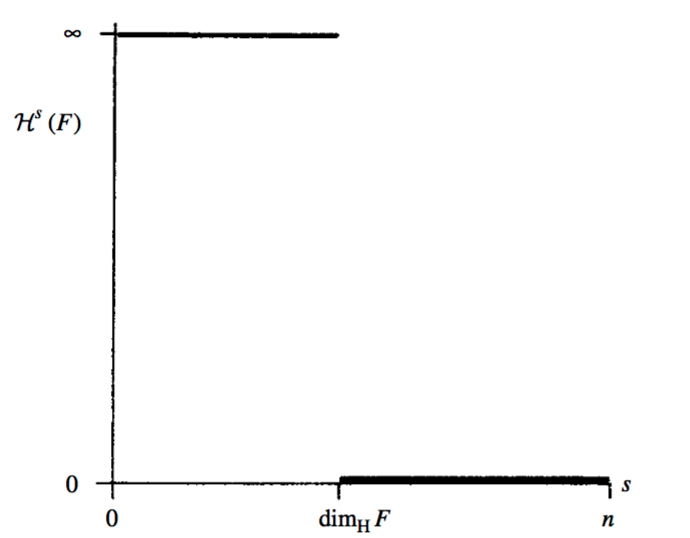
In particular, the above equation holds for any differentiable function with bounded derivative; such a function is necessarily Lipschitz as a consequence of the mean value theorem. If f is an isometry, i.e. , then . Thus, Hausdorff measures are translation invariant (i.e. , where }, and rotation variant, as would be certainly be expected. [Taken from 1, Page 28-30]

**Hausdorff Dimension –** “Returning to equation (1), it is clear that for any given set and , is non-increasing with , so by (2) is also non-increasing. In fact, rather more is true: if and is a -cover of we have

so taking infima, . Letting we see that if then for . Thus a graph of against in the figure below shows that there is a critical value of at which ‘jumps’ from to . This critical value is called the of , and written ; it is defined for set . (Note that some authors refer to Hausdorff dimension as Hausdorff-Besicovitch dimension.) Formally

(taking the supremum of the empty set to be 0), so that

If , then may be zero or infinite, or may satisfy



Graph of against for a set . The Hausdorff dimension is the value of at which the ‘jump’ from to

A Borel set satisfying this last condition is called an -. Mathematically, - are by far the most convenient sets to study, and fortunately they occur surprisingly often.

**Calculation of Hausdorff dimension**

**Example 1 [Taken from 1, Page 34] .** *Let be the Cantor dust constructed from the unit square as in figure below. (At each stage of the construction the squares are divided into 16 squares with a quarter of the side length, of which the same pattern of four squares is retained.)*

*Then so*

**Calculation.** Observe that the th stage of the construction, consists of squares of side and thus of diameter . Taking the squares of as a -coverof where , we get an estimate for the infimum in (1). As so giving .

For the lower estimate, let proj denote orthogonal projection onto the -axis. Orthogonal projection does not increase distances, i.e. if , so proj is Lipschitz mapping. By virtue of the construction of , the projection or ‘shadow’ of on the -axis, proj, is the unit interval [0,1]. Using (4),

.

**Example 2 [Taken from 1, Page 34]** *Let be the middle third Cantor set (see figure on first page). If then and .*

**Calculation –** We call the intervals that make up the sets in the construction of -. Thus ­ consists of level-k intervals each of the length .

Taking the intervals of as a -cover of gives that if . Letting gives .

To prove that we show that

for any cover of . Clearly, it enough to assume that the are intervals, and by expanding them slightly nd using compactness of , we need only verify the equation above if is a finite collection of closed subintervals of . For each , let be the integer such that

.

Then can intersect at most one level- interval since the separation of these level- is at least . If then, by construction, intersects at most level- intervals of using the equation above. If we choose large enough so that for all, then, since the intersect all basic intervals of length counting intervals gives which reduces to .

**Some Alternative Definitions of Dimension**

**Self-similarity Dimension –** “*If a set in can be subdivided into some finite number*  *of subsets, all congruent (by translations or rotations) to one another and each a rescaled copy of by a linear factor , then the “self-similarity dimension” of is the unique value d that satisfies , i.e.*

”[5, Page 512]

**Minkowski–Bouligand Dimension** **(Box-Counting Dimension) –** “*Suppose that N(ε) is the number of boxes of side length* ε *required to cover the set. Then the box-counting dimension is defined as*

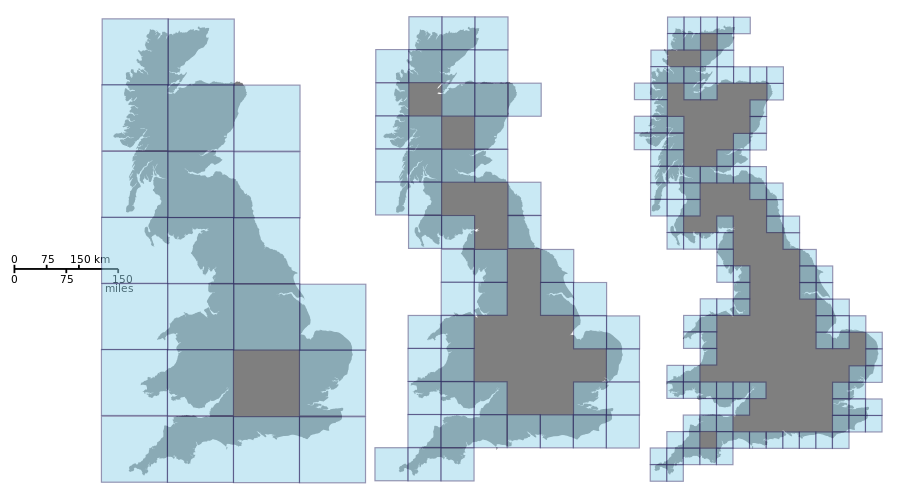
*­­*

.” [6]

Estimating the Coast line of Britain.

1.21

Image taken from [6]



**SELF SIMILARITY**

**Definition.** “A subset (a metric space) is called an *ε-net* in if for every , there is a point s such that ε.” [Copied from 4, Page 8]

**Theorem on ε-nets.** “A metric space , is compact iff it is complete and for every , there is a finite ε-net in M.” [Copied from 4, Page 8]

**Proof of** Assume that is compact. Consider , a Cauchy sequence, in We have to show that it has a limit. M is compact, hence the sequence contains a convergent subsequence . Let . Indeed, since is a Cauchy sequence, such that . Also, , such that . So, for we have

Therefore, .

Assume that for some , there is no finite ε-net in M. Then I claim that M contains a sequence with the property that

(5)

Indeed, we construct the desired sequence by induction. Choose arbritrarily. Suppose that points satisfying (5) are already chosen. Since, the finite set is not an ε-net, there is a point such that ε for .

The sequence satisfying the above equation does not contain any convergent subsequence, because every subsequence also satisfies (5), hence is certainly not a Cauchy sequence. A contradiction. Thus, M is complete.

2. Assume that is complete and that , a finite ε-net in . Let us show that M is compact, that is, that every sequence contains a convergent subsequence. It is enough to find in a Cauchy subsequence. Let be a finite ε-net in for . Denote the points of by , where is the number of points ,

Denote the closed ball of radius centred at . Since is ε-net in M, the union of the balls , , covers the whole set . Put . So at least one of the balls , , contains infinitely many terms of our sequence. Therefore, exists an infinite subsequence, that is contained in a ball of radius 1.

Now put . At least one of the balls contains infinitely many terms of the subsequence . Hence, a subsequence that is contained in a ball of radius . And so on. Consider the diagonal subsequence . This is the desired Cauchy sequence, because its terms, starting with the th, beling to a ball of radius. [Similar to 4, Page 8-9]

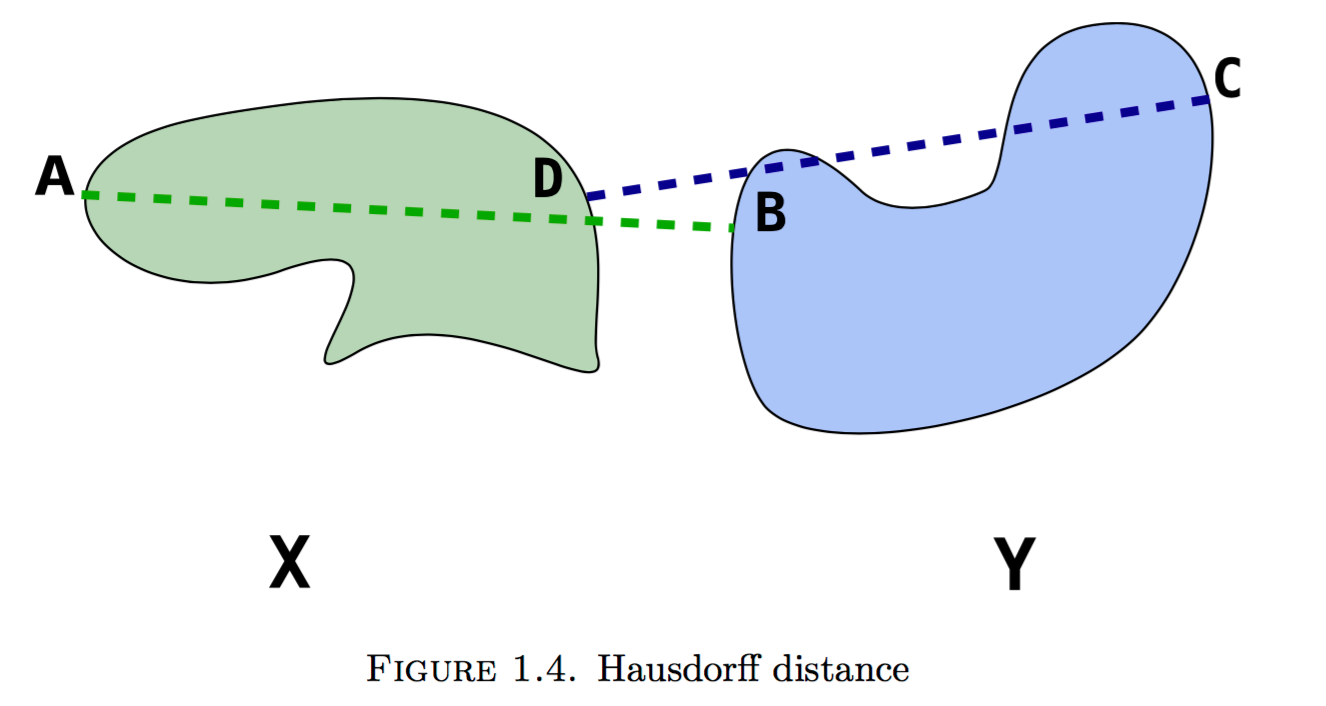
**Definition.** “Let be a metric space. We denote by the collection of all nonempty compact subset of . We want to define a distance between two compact sets so that is itself a metric space. For this, we define first the distance between a point and a compact set

Now the distance between two sets and is defined by

A more detailed expression for the same distance is

.” [Copied from 4, Page 10]

**Definition.** “Let be a metric space. A map f of some subset to is called an *-perturbation* if for all .” [Copied from 4, Page 11]



is called the Hausdorff Distance. Image from [4, Page 10]

“Then the statement “the Hausdorff distance between X and Y is d” is equivalent to the statement “there exists an -perturbation and an -perturbation such that ” [4, Page 11]

**Theorem.** “If the metric space M is complete (resp. compact), the is complete (resp. compact) as well.” [4, Page 11]

Assume now that a family of contracting maps in is given. Define the transformation

**Theorem.** “The map is contracting. Therefore, if is complete, there is a unique nonempty compact subset satisfying .” [Copied from 4, Page 11]

**Definition.** “The set from the theorem above is called a ***homogeneous self-similar fractal set*** . The system of functions is usually called an *iterated function system (i.f.s*  for short) defining the fractal set .” [Copied from 4, Page 11]

Sometimes a more general definition is used. Namely, instead of the (6), let us define the map by the formula

where Y is a fixed compact subset of M. [4, Page 11]

**Definition.** “A set that is fixed point for the map given above is called an ***inhomogeneous self-similar fractal set*** .” [Copied from 4, Page 11]

Example taken from [4, Page 12] **Cantor Set –** . Here , . It is instructive to look at how , the fixed point for, is approximated by a sequence of sets defined by the recurrence .

Choose first ; then

The sequence is decreasing, , and the limit set is .

Now put

, ….

The sequence { is increasing, , and the limit set is the closure of . Note that is not compact. Therefore, it is not a point of .”

Example taken from [4, page 13]. **Sierpinski Gasket –** Here , the complex plane. Let be the sixth root of 1. Define

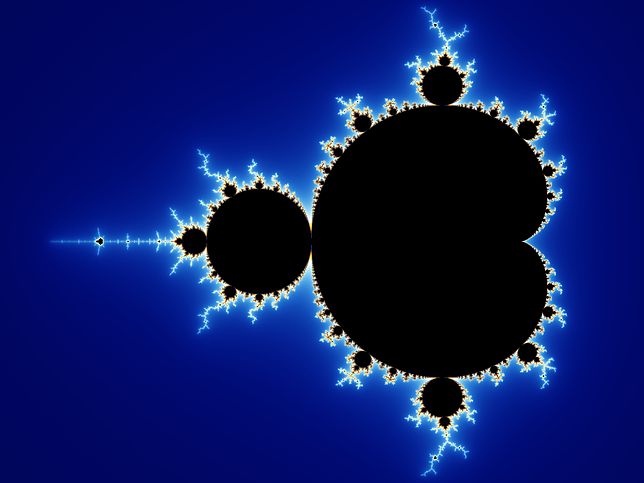
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The fractal defined be the i.f.s {, is called a *Sierpinski Gasket.*

In this case, these are three natural choices for the initial set . First, take as the solid triangle with the vertices Then the sequence is decreasing and . Second, we let be the hollow triangle with vertices at 1. Then the sequence ) is increasing and is the closure of .

**SOME OTHER IMPORTANT FRACTALS**

**Mandelbrot Set –** *The Mandelbrot Set is the set of complex numbers for which the function does not diverge when iterated from , i.e. for which the sequence ,….. remains bounded in absolute value.*  [7]

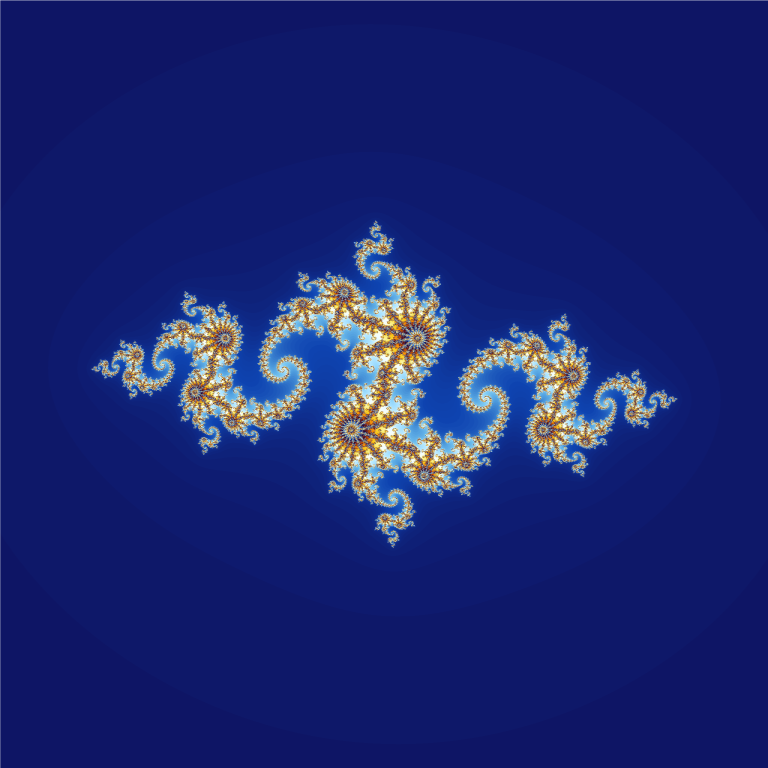
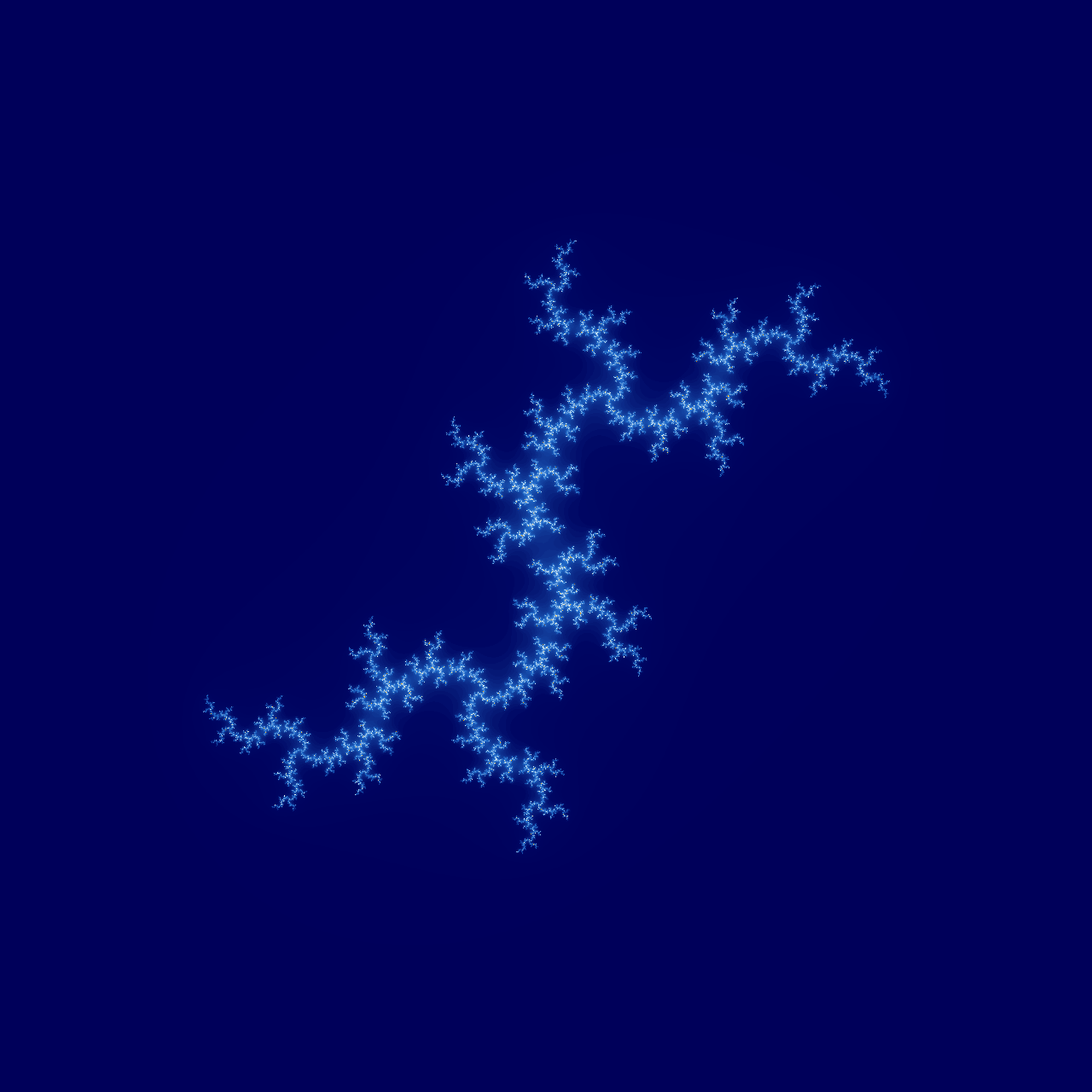


Initial image of a Mandelbrot set zoom sequence with a continuously colored environment. Non black points correspond to numbers that are outside the set. Image taken from [7]

**Fun Fact -** !

**Filled-in Julia Sets –** “*The filled in Julia set of the function is defined as*

.” [8]

First Image : , where , where is the golden ratio.

Second Image : , where

Third Image :, where

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1. (“The class of Borel Sets is the smallest collection of subsets of with following properties: (a) every open set and every closed set is a Borel set (b) the union of every finite or countable collection of Borel Sets is a Borel set, and the intersection of every finite or countable collection of Borel sets is a Borel set.” [2, Page 6]) [↑](#footnote-ref-1)